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LSF modelling with zero to many parameters

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Abstract

This note expands the minimum-dimension LSF modelling (GAIA–C3–TN–LU–LL–084–01) with a few more useful ideas: (1) that the LSF model is formulated such that the area normalization is implicit; (2) that the non-negativity of the LSF can be guaranteed by simple constraints on the parameters; and (3) that the model should work also with very few parameters, even without any parameters at all.

Document History

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1 Introduction

A previous technical note (GAIA–C3–TN–LU–LL–084–01: *Minimum-dimension LSF modelling* (Lindegren, LL-084)) described how the Line Spread Function (LSF) can be expanded in basis functions that were derived by Principal Component Analysis (PCA) of a large ensemble of randomly generated, but physically plausible LSFs. Sorting the basis functions by importance (as measured by the singular values of the dispersion matrix) allows to define the smallest set of basis functions that are able to represent typical LSFs to a given precision. The LSF is written as a linear combination of the basis functions, and the coefficients are the model parameters. The note also suggested a way to represent the basis functions by means of a spline function with analytical tails.

A few important questions were left open in Lindegren (LL-084). Any LSF model $L(u)$ should satisfy two constraints: non-negativity ($L(u) \geq 0$ for all u), and unit area ($\int_{-\infty}^{+\infty} L(u)\,\mathrm{d}u = 1$). The representation in Lindegren (LL-084) does not guarantee non-negativity, and unit area is introduced as an explicit constraint on the model parameters (Eq. 3 in Lindegren (LL-084)). Moreover, the actual choice of model parameters was rather vague, partly because of the normalization problem.

In this note I propose a very simple extension of the model which eliminates the normalization constraint, gives some handle on the non-negativity issue, and provides a clear proposal for the choice of model parameters. The resulting model is quite general and not restricted to the particular numerical/analytical representation described in Lindegren (LL-084).

2 Expansion of the LSF in basis functions

This section recalls the formulation in Lindegren (LL-084). Based on Eq. (5) of that note, but omitting the tilde on L and redefining the meaning of N, the LSF model can written in terms of the basis functions $B_n(u)$ as

$$L(u) = \sum_{n=0}^{N-1} c_n B_n(u - \delta u)\,, \tag{1}$$

where the shift δu is a model parameter defining the origin of the LSF (Lindegren, LL-080). It should be noted that none of the basis functions corresponds to a (small) shift of the profile, so δu is an independent model parameter; it is also necessary for representing, e.g., chromaticity. This model has $N+1$ parameters (δu, c_0, c_1, $\ldots$, c_{N-1}). However, there are only N free parameters, since the coefficients must satisfy the unit area constraint

$$\sum_{n=0}^{N-1} b_n c_n = 1\,, \tag{2}$$

where

$$b_n = \int_{-\infty}^{+\infty} B_n(u)\,\mathrm{d}u\,. \tag{3}$$

An example of the first 13 basis functions $B_0(u)$ through $B_{12}(u)$ was shown in Figs. 4–6 of Lindegren (LL-084). $B_0(u)$ is special, because it is simply the ensemble mean LSF; it is also strictly positive everywhere, but not accurately normalized as derived in Lindegren (LL-084), i.e., $b_0 \neq 1$ in general.

3 Normalization of the basis functions

Let us define an equivalent set of 'normalized' basis functions $H_n(u)$ by the following process:

$$H_0(u) = \frac{1}{b_0} B_0(u)\,, \tag{4}$$

$$\left.\begin{aligned} \bar{B}_n(u) &= B_n(u) - b_n H_0(u) \\ H_n(u) &= \frac{\bar{B}_n(u)}{\max\limits_{-\infty<x<+\infty} \left|\bar{B}_n(x)/H_0(x)\right|} \end{aligned}\right\} \quad n = 1,\, 2,\, \ldots,\, N-1\,. \tag{5}$$

Equation (4) ensures that $H_0(u)$ is normalized to unit area. The first line of Eq. (5) ensures that the integral of any of the functions $\bar{B}_n(u)$, $n > 0$, is exactly zero. The second line of Eq. (5) ensures that $|H_n(u)| \leq H_0(u)$ for all values of u. The intermediate functions $\bar{B}_n(u)$ are not further used. The new basis functions $H_n(u)$, $n = 0,\, 1,\, \ldots,\, N-1$ have the following useful properties:

- since they are linearly independent by construction, they span the same linear space as the original basis functions;
- the integral of any linear combination $\sum_{n=0}^{N-1} h_n H_n(u)$ is equal to h_0;
- for any u we have the inequality

$$\left| \sum_{n=1}^{N-1} h_n H_n(u) \right| \leq H_0(u) \sum_{n=1}^{N-1} |h_n|\,. \tag{6}$$

The first property means that any LSF that can be represented by the original basis functions can also be represented by the normalized set, i.e., in the form

$$L(u) = \sum_{n=0}^{N-1} h_n H_n(u - \delta u)\,. \tag{7}$$

The second property means that unit area can be ensured simply by fixing $h_0 = 1$, thus eliminating one model parameter and one constraint. The third property means that non-negativity of the LSF is guaranteed if

$$\sum_{n=1}^{N-1} |h_n| \leq 1 \, . \tag{8}$$

A simpler but more restrictive condition for non-negativity is obviously

$$|h_n| \leq \frac{1}{N-1} \, , \quad n = 1, \, 2, \, \ldots, \, N-1 \, . \tag{9}$$

In practice, non-negativity can *almost* be ensured even if these limits are exceeded, especially if N is greater than a few.

However, it is not expected that any of these constraints is actually implemented in the LSF calibration, as they may be too restrictive in any particular case. It is probably better to check the non-negativity of the estimated LSF, and correct if necessary (perhaps by adjusting the 'guilty' coefficient).

The normalization in Eq. (5) is nevertheless useful, as it provides some absolute meaning to the sizes of the coefficients: a coefficient whose absolute value is of the order of 1 [or $(N-1)^{-1}$, or perhaps $(N-1)^{-0.5}$] could be considered as 'large'.

4 Choice of model parameters

Since $h_0 = 1$ is no longer a free parameter, we can use h_0 to denote the shift of origin (instead of δu). The complete model, with N free parameters h_n, $n = 0, \, \ldots, \, N-1$, is then:

$$L(u) = H_0(u - h_0) + \sum_{n=1}^{N-1} h_n H_n(u - h_0) \, . \tag{10}$$

In the case of $N = 0$ model parameters (or if all the parameters are zero) it reduces to the default (*a priori*) model $L(u) = H_0(u)$. Non-negativity of $L(u)$ is strictly guaranteed if

$$\sum_{n=1}^{N-1} |h_n| \leq 1 \, , \tag{11}$$

although in practice the constraint can probably be somewhat relaxed if $N > 2$.

Although we arrived at the model in Eq. (10) via the particular basis functions derived in Lindegren (LL-084), the resulting model is more general: it can be applied to many other analytical and numerical expansions of the LSF where the first component (H_0) has unit area, and all subsequent components have zero area, provided that none of the components corresponds to a small shift of the LSF (effectively this means that $H_0'(u)$ cannot be represented in this basis). For some of them the normalization in the second line of Eq. (5) can be used to define non-negativity conditions similar to the present model.

5 References

[LL-080], Lindegren, L., 2009, *A framework for consistent definition and use of LSFs in AF/BP/RP/RVS*,
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