Local plane coordinates for the detailed analysis of complex Gaia sources

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Abstract. The detailed analysis of a complex object, such as a partially resolved visual binary, may be simplified if the observations are referred to a local coordinate system after taking into account the satellite attitude, instrument geometry, aberration and gravitational deflection. Thus these effects need not be further considered in the analysis. This is possible, to microarcsec accuracy, within a radius of $\simeq 1$ arcmin from a chosen reference point. The definition of such local plane coordinates (LPCs) is given, together with a specification of the data that need to be provided with the individual observations. The advantage of the method is that the object can be analyzed without access to attitude, geometric calibration and orbit data, and without need for the corresponding transformations. The necessary computations will probably be done by CU4 and CU5 "on the fly", in order to avoid extra inter-CU data flow volume.

The second issue of GAIA-LL-061 made the concept more complete compared to issue 1 of 9 October 2005. The third issue pointed out that the aberrational contraction/expansion of the sky is automatically taken into account by the proposed computational procedure, and it added the formulae for the computation of the scan direction. The fourth issue corrected two little errors discovered by Craig Stephenson. The fifth issue introduces the barycentric correction (Roemer delay) in (4) and the use of t_B (the barycentric date) as the time argument for the LPCs. The sixth issue contains the new Appendix C detailing the computation of the LPC centoids (epoch astrometry). The seventh issue changes the sign convention for the barycentric correction (see footnote 8) and adds Appendix D on the transformations for changing the reference point. Issue 8 adds Appendix E with an example where the standard stellar model is fitted to the epoch astrometry for Barnard's star.

The source files for this document are in the DPAC Subversion repository under `http://gaia.esac.esa.int/dpacsvn/DPAC/CU3/docs/General/LPC-LL-061`

1 Introduction

The detailed analysis of window (sample) or elementary (centroid) data for a non-single object is complicated by the fact that the individual observations only make sense in relation to certain attitude, calibration, and orbital data. These auxiliary data may be given in several separate, possibly quite large data sets, and their proper use requires a long sequence of transformations that are not really central to the specific object analysis problem at hand.

At the level of the object processing (CU4) and 2-d imaging (CU5), these auxiliary data are known from the astrometric AGIS and other core processes, and it would seem like a good idea to 'correct' the window/elementary observations for these (known) effects, before handing them over to the object processing. This note describes a possible procedure for this. If and how it should actually be used is briefly discussed in Sect. 6.

The basic idea is to define for each object a local plane coordinate (LPC) system in the tangent plane of the unit sphere, in which the *coordinate direction* of the object is modelled. The local system is uniquely defined by the chosen celestial coordinates of the tangent point (α_0, δ_0). The local coordinates are similar to the 'standard coordinates' customarily used in small-angle astrometry (e.g., [2], [4]), basically obtained through a gnomonic (central) projection onto the tangent plane. For the individual observations, data are referred to axes oriented according to the local scan direction.

An important constraint for such local coordinates to be practically useful is that the differences between coordinate directions and observed (proper) directions are locally negligible after allowing for a change of origin and scale. This limits the size of the area in which local coordinates can be used, as discussed in the next section.

2 Differential effects of aberration and light deflection

To first order, stellar aberration causes the apparent position of an object to be displaced by the small angle $(v/c)\sin\psi$ towards the direction of motion (apex), where v is the speed of the observer, c the speed of light, and ψ the angle of the object from the apex point. It is easily seen that, within a small area around the object, the differential effect amounts to an isotropic change of angular scale by the factor $1-(v/c)\cos\psi$. Thus, an object located near the apex appears smaller by the (linear) factor $1-(v/c)$, while an object near antapex appears magnified by the factor $1+(v/c)$. Since the differential effect is isotropic, there is no distortion of the image: a circular object remains circular.

The barycentric velocity of Gaia is always around 30 km s^{-1}, or $v/c \simeq 10^{-4}$. The apparent magnification factor consequently varies between 0.9999 and 1.0001. At Gaia accuracies, this is significant even for 'small' objects; e.g., in a binary with component separation 1 arcsec, the apparent separation may vary by ± 100 μas (it also affects the apparent flux by some fraction of a millimag). Thus, e.g., if the object *appears* smaller, the *observed* proper direction offsets have to be *increased* in the transformation to coordinate direction offsets.

On a much larger angular scale, the differential effect obviously cannot be isotropic, so the representation by a single magnification factor breaks down at a certain radius. From numerical tests using the full relativistic aberration formula (Lorentz transformation), it is found that, for $v = 30$ km s^{-1}, the simple magnification model is accurate to ≤ 1 μas within a radius of $\simeq 1$ arcmin. This holds even for the most unfavourable case of $\psi \simeq 90°$. The residual error increases quadratically with the radius.

There are other effects besides aberration that must be considered when defining the local plane coordinates, the most important being gravitational deflection by the Sun. This causes an apparent shift by approximately (4 mas) $\cot(\psi/2)$ away from the Sun, where ψ is now the angle of the object from the Sun. In contrast to the aberration, the effect is anisotropic: the image is always compressed along the circle through the object and the Sun, and somewhat less compressed (or, for $\psi < 90°$, magnified) in the perpendicular direction. Thus, differential light deflection cannot be described by a single scale factor. Fortunately, the effect is much smaller than for the aberration, and does not exceed 1 μas within a radius of 19 arcsec in the most unfavourable case of $\psi = 45°$ (the smallest Sun angle allowed by the scanning law). The residual effect increases linearly with radius, and may in practice be neglected (< 3 μas) within a radius of 1 arcmin.

To sufficient accuracy for the Gaia object processing, the differential effects of aberration and light deflection by the Sun can therefore always be represented by a simple change of scale within a field of up to ~ 1 arcmin radius.

There are exceptional circumstances where this model is inadequate to describe the local distortion. This may be the case near one of the major planets (Jupiter, Saturn, etc.). A possible solution could be to model the gravitational deflection near these bodies as a superposed, local effect. Moreover, the local coordinates are of course only useful for objects that stay within the 1 arcmin area for the duration of the mission; this applies to all stellar and extragalactic objects, but not to solar-system objects.

The use of local plane coordinates, in which differential aberration is accounted for by a change of scale, was originally adopted for the NDAC processing of double stars [3].

3 Transformations involving local plane coordinates

For the reference point (α_0, δ_0) we define the reference triad $[\,\boldsymbol{p}_0\ \boldsymbol{q}_0\ \boldsymbol{r}_0\,]$ by means of the three orthogonal unit vectors

$$\boldsymbol{p}_0 = \begin{bmatrix} -\sin\alpha_0 \\ \cos\alpha_0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{q}_0 = \begin{bmatrix} -\sin\delta_0\cos\alpha_0 \\ -\sin\delta_0\sin\alpha_0 \\ \cos\delta_0 \end{bmatrix}, \qquad \boldsymbol{r}_0 = \begin{bmatrix} \cos\delta_0\cos\alpha_0 \\ \cos\delta_0\sin\alpha_0 \\ \sin\delta_0 \end{bmatrix}. \tag{1}$$

$\boldsymbol{r}_0$ is the direction to the reference point, while $\boldsymbol{p}_0$ indicates local 'East' (increasing α) and $\boldsymbol{q}_0$ local 'North' (increasing δ) if $|\delta_0| < 90^\circ$. Note however that the reference triad is perfectly well-defined also at the celestial poles, where α_0 remains significant.[1]

For an arbitrary coordinate direction $\boldsymbol{c}$ (in the vicinity of $\boldsymbol{r}_0$, so that $\boldsymbol{r}_0'\boldsymbol{c} > 0$) local plane coordinates (LPC) (a, d) are usually defined through gnomonic projection[2]

$$a = \frac{\boldsymbol{p}_0'\boldsymbol{c}}{\boldsymbol{r}_0'\boldsymbol{c}}\,, \qquad d = \frac{\boldsymbol{q}_0'\boldsymbol{c}}{\boldsymbol{r}_0'\boldsymbol{c}}\,. \tag{2}$$

The inverse relation is

$$\boldsymbol{c} = \frac{\boldsymbol{r}_0 + \boldsymbol{p}_0 a + \boldsymbol{q}_0 d}{(1 + a^2 + d^2)^{1/2}}\,. \tag{3}$$

Equations (2) and (3) are formally identical to the usual transformations involving standard coordinates (e.g., [4]), except for the purely practically motivated absence of the denominators. However, we make the important distinction that $\boldsymbol{c}$ must always be interpreted as a *coordinate direction*, not as an observed (proper) direction. This guarantees that the object modelling can be completely carried out in (a, d). For example, the combination of proper motion, parallax and a Keplerian orbit (using Thiele–Innes elements) can be parameterized as

$$\left.\begin{aligned} a &= a_T + (t_\mathrm{B} - T)\mu_{\alpha*} + f_a(t)\varpi + BX(t_\mathrm{B}) + GY(t_\mathrm{B}) \\ d &= d_T + (t_\mathrm{B} - T)\mu_\delta \;\; + f_d(t)\varpi + AX(t_\mathrm{B}) + FY(t_\mathrm{B}) \end{aligned}\right\} \tag{4}$$

where (a_T, d_T) is the offset at epoch T and f_a, f_d are the known parallax factors, etc; t is the coordinate time (TCB) of the observation, and t_B is the time of observation corrected for the Roemer delay,

$$t_\mathrm{B} = t + \boldsymbol{r}_0'\boldsymbol{b}_\mathrm{G}(t)/c\,, \tag{5}$$

where $\boldsymbol{b}_\mathrm{G}(t)$ is the barycentric position of Gaia at the time of observation and c is the speed of light. The use of t_B as the time argument in (4), rather than t, eliminates the need for the user to consider the Roemer delay when modelling the motion in the (a, d) coordinates. As indicated in the equation, this applies both to the proper motion effect and any orbital motion.[3] (As indicated in the equations above, the parallax factors and

[1] For example, the two reference points $(\alpha_0, \delta_0) = (0^\circ, 90^\circ)$ and $(\alpha_0, \delta_0) = (90^\circ, 90^\circ)$ are not equivalent: they define different reference triads.

[2] The usual notation for standard coordinates is (ξ, η), but since η is used for the along-scan field angle, (a, d) is here suggested for the local plane coordinates. a and d roughly correspond to $\Delta\alpha\cos\delta$, $\Delta\delta$.

[3] The barycentric correction is important for sources with a high proper motion, or a large orbital motion. The maximum difference $|t_\mathrm{B} - t| \simeq 500$ s, corresponding to the light time of 1 au, during which Barnard's star moves about 160 μas. – In (5) the barycentric correction is calculated for the fixed direction $\boldsymbol{r}_0$, whereas it should, more correctly, be calculated using the coordinate direction $\boldsymbol{c}$ at the time of the observation. Within a radius of 1 arcmin from the reference point, the maximum error from this approximation (for Barnard's star) is only (160 μas) $\times \sin 1' \simeq 0.05$ μas, so the approximation is acceptable which greatly simplifies the use of the LPCs.

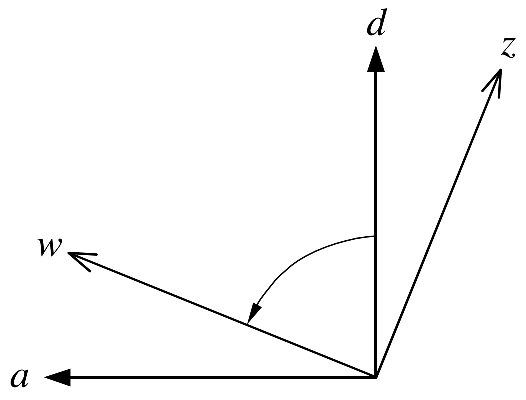

FIGURE 1: The transformation between local celestial coordinates (a, d) and local scan coordinates (w, z) is uniquely given by the position angle of the scan, θ. The figure indicates the sense of directions as seen from the centre of the celestial sphere.

$\boldsymbol{b}_{\mathrm{G}}$ are evaluated at t, not t_{B}. But because this is only done when the LPCs are generated, the subsequent user of the LPCs has no need for t.)

In order to express observations as simply as possible in the local system, we make however one further transformation, viz., from the local *celestial* coordinates (a, d) to the local *scan* coordinates (w, z) (Fig. 1). This transformation is completely determined by θ, the position angle of the scan:

$$\left.\begin{aligned} w &= a\sin\theta + d\cos\theta \\ z &= -a\cos\theta + d\sin\theta \end{aligned}\right\} . \tag{6}$$

Conversely,

$$\left.\begin{aligned} a &= w\sin\theta - z\cos\theta \\ d &= w\cos\theta + z\sin\theta \end{aligned}\right\} . \tag{7}$$

Loosely speaking, $+w$ is the local direction in which the FOV moves on the sky (the AL direction in the local system), while $+z$ is the local AC direction (with the Sun at $z > 0$). The precise definition, however, is the following: $+w$ is the local direction of increasing field coordinate η; $+z$ is the local direction of increasing field coordinate ζ. The procedure for calculating θ for a particular observation is detailed in Appendix B. It involves nothing but the attitude and the instantaneous field coordinates (η, ζ) of the celestial object under consideration.

The use of (w, z) instead of (a, d) allows to distinguish easily between AL and AC quantities. For example, in many cases only the AL coordinate is of interest, and we may then omit z altogether. Also, the observational errors are normally uncorrelated between w and z (and usually much larger in z than in w), whereas a strong correlation would usually be found between a and d. The specification of samples, patches and windows is also naturally related to (w, z).

The transformation of an arbitrary object model to the local scan coordinates is straightforward; for example, for the model in (4) we obtain

$$\left.\begin{aligned} w &= sa_T + cd_T + (t_\mathrm{B} - T)s\mu_{\alpha*} + (t_\mathrm{B} - T)c\mu_\delta + f_w\varpi + XcA + XsB + YcF + YsG \\ z &= -ca_T + sd_T - (t_\mathrm{B} - T)c\mu_{\alpha*} + (t_\mathrm{B} - T)s\mu_\delta + f_z\varpi + XsA - XcB + YsF - YcG \end{aligned}\right\} \tag{8}$$

where, for brevity, we have put $s = \sin\theta$, $c = \cos\theta$ and introduced the parallax factors in local scan coordinates,

$$\left.\begin{aligned} f_w &= f_a \sin\theta + f_d \cos\theta \\ f_z &= -f_a \cos\theta + f_d \sin\theta \end{aligned}\right\} . \tag{9}$$

4 Expressing observations in local coordinates

4.1 Elementary observations (centroid positions)

The outcome of the image centroiding process is a determination of the accurate time t when the image centre crossed an abstractly defined fiducial line on the respective CCD detector, interpolated to sub-pixel resolution (note that the observations are labelled with the instant when a specific sample line of the corresponding CCD window was was transferred to the read-out register, the difference being half the exposure time on the CCD). This is the main along-scan (AL) astrometric observation. There is also an across-scan (AC) coordinate, given by the CCD number (n) and pixel column number (m) [1], which for the SM will also always be interpolated to sub-pixel resolution. For the AF it can be measured only in exceptional cases (bright stars and special calibration modes) due to the on-chip binning in the AC direction. The geometrical calibration of the CCDs provides the mapping from (n, m) and FOV index to field angles (η, ζ), which combined with the attitude gives the observed (proper) direction $\boldsymbol{u}$ to the object at the instant t. Removing aberration and gravitational light deflection gives the coordinate direction $\boldsymbol{c}$ at time t, from which (a, d) are computed by means of (2) and finally (w, z) by (6).[4]

Note that this computational procedure automatically takes into account (i.e. corrects) the aberrational contraction/expansion of the sky discussed in Section 2.

Note furthermore that for the above transformations the knowledge of the across-scan field coordinate ζ is needed. This cannot be derived directly for the vast majority of the AF

[4] This computation can be simplified by skipping the intermediate coordinates (a, d)

measurements. In these cases it must thus either be extrapolated from the corresponding measured SM centroid or from prior knowledge of the object's astrometric parameters. Appendix A shows that integer-pixel precision is not sufficient for the purpose.

Now let $\sigma_w = \sigma_\eta$ and $\sigma_z = \sigma_\zeta$ be the standard errors of the centroiding AL and AC, respectively, expressed as angles; then the complete specification of the elementary astrometric observation is: $(t, \theta, w, \sigma_w, z, \sigma_z, f_w, f_z)$, where z, σ_z can be omitted for a purely one-dimensional (AL) observation (at least σ_z will not make much sense). Note, however, that we added the parallax factors (f_w, f_z), so that one does not need a satellite ephemeris to interpret the observations. In addition, the reference point (α_0, δ_0) must of course be specified; this would be the same for all observations of a given object.

We herewith define the reference point (α_0, δ_0) to be the catalogue position (i.e. the position at the catalogue epoch) in the astrometric source catalogue file belonging to the same delivery of the Gaia Main Database as the local plane coordinates file under consideration. This definition is always unambiguous, and it is in accordance with the 'versioning' concept for the overall data flow in the Gaia data reduction. It also does not entail any extra data flow or data organization, since the astrometric parameters of any object will always be needed for object processing.

In the original data (t, n, m), the AL positional information was essentially provided by the time t, which therefore had to be given with a resolution of some nanoseconds.[5] When transformed to local scan coordinates, the AL positional information is instead given by w, and t is only needed to the moderate precision determined by the object's motion or variability. For example, a common value for t may suffice for each FOV transit (not for planetary objects!), and the sequence of (w, z) values for the individual CCD transits could similarly be condensed, by averaging, to a single observation for each FOV transit.[6] Table 2 shows a possible specification of an LPC file, i.e. of the input data for the "object processing" by CU4 per celestial source.

Note that photometric data are not included in the table, although it would be natural to include at least the G magnitudes (calibrated, and probably averaged over all AF strips) and some RP/BP colour information for each FOV transit in the same structure.

[5]Double precision is not sufficient to express time to that resolution over a sufficiently long period. A suitable format for absolute time scales, and the one chosen by DPAC, is instead be the number of nanoseconds from J2010.0, expressed as a signed long integer (64 bits). The wrap-around time is 2^{63} ns = 292 years.

[6]During the discussions of issue 1 of the present document, both D. Pourbaix (for CU4) and U. Bastian (for CU3) have argued against this averaging, because it would kill a lot of noise information and outlier treatment possibilities (cosmic rays, disturbing stars from the other field of view and so on). It also would disable CU4 to independently take the correlations between different CCD transits from the same field-of-view transit into account in their error calculus. L. Lindegren has argued to the contrary, saying that all this can be done by CU3 already, in the computation of the average and its standard error. The simplifications for the later stage (object processing) are very great, indeed. This discussion is still open. It has no impact on data volumes if CU4 produces the LPCs "on the fly".

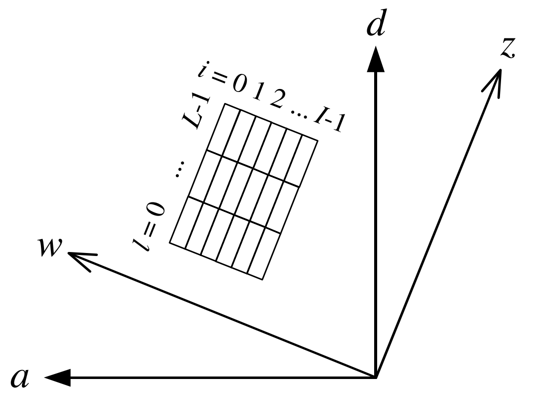

FIGURE 2: Specification of a window of $I \times L$ samples in local scan coordinates (w, z).

This inclusion was indeed requested by CU4 during the discussion of version 1 of the present document. Of course, CU4 is well justified to request a smooth and easy-to-use combination of astrometric and photometric data. However, the present document intends to specify the conceptual interface between Gaia's core astrometry and its users. The combination to other data should be an issue in the compilation of the Interface Control Document for the Gaia Main Database, into which the whole concept of the present document has to enter eventually, too.

4.2 Window data (samples), simplified model

The local scan coordinates are convenient also for specifying in an absolute sense the celestial location of any sample, patch or window in the CCD data stream. Consider the fairly general case of a window containing $I \times L$ samples, as shown in Fig. 2 (one-dimensional windows are represented with $L = 1$). Within the window, let the samples be indexed $i = 0 \dots (I - 1)$ along scan and $l = 0 \dots (L - 1)$ across scan.

For ease of reference and ease of understanding we give the original model of issue 1 in the present subsection, and a more complete one in the following subsection.

Taking the sample indexed $(i, l) = (0, 0)$ as origin, we may compute the observed (proper) direction corresponding to its centre exactly as for the image centroid in Sect. 4.1. Removing aberration and light deflection gives the coordinate direction and hence, using (2) and (6), its local scan coordinates (w_0, z_0). In principle we could repeat the calculation for each sample, but it is sufficient to compute the sample dimensions $(\Delta w, \Delta z)$ (which may be negative depending on the adopted indexing convention), from which

$$\left.\begin{aligned} w_i &= w_0 + i\Delta w \quad (i = 0 \ldots I-1) \\ z_l &= z_0 + l\Delta z \quad\ \ (l = 0 \ldots L-1) \end{aligned}\right\} . \tag{10}$$

A simple expedient (albeit perhaps not the most efficient one) could be to compute (w_{I-1}, z_{L-1}) rigorously as for the first sample, and then use (10) to compute $(\Delta w, \Delta z)$. It should be noted that the sample dimensions vary among the observations because of differential optical distortion and differential aberration, so it will not be possible to use fixed values.

The complete specification of the window data would consist of t (the approximate time of observation); θ (the position angle of the scan); I and L (the size of the window); w_0 and z_0 (the local scan coordinates of the first sample); Δw and Δz (the sample dimensions in local scan coordinates); and the sample values S_{il} for $i = 0 \ldots I-1$ and $l = 0 \ldots L-1$. Again, the parallax factors f_w and f_z should be added, but they are in practice identical for all CCD transits in a given FOV transit. Table 3 shows a possible specification of the input data per object, neglecting additional information needed for photometric uses, but already including the shear terms introduced and motivated in the following subsection.

4.3 Window data (samples), complete model

The previous subsection mentions optical distortion and differential aberration to cause variations in the effective sample dimensions $(\Delta w, \Delta z)$. In fact there are more causes for such variations: Focal-length evolution, scan motion variations along scan, non-zero attitude motion across-scan. Furthermore, most of these effects do not only create changes in $(\Delta w, \Delta z)$, but they also lead to shear terms in (10). These shear terms are too large to be ignored, as will be shown below. A complete model for the window (sample) data would thus have to use an extension of (10) in the following form:

$$\left.\begin{aligned} w_{i,l} &= w_{0,0} + i\Delta w + lc_w \quad (i = 0 \ldots I-1, l = 0 \ldots L-1) \\ z_{i,l} &= z_{0,0} + l\Delta z + iv_z \quad\ \ (i = 0 \ldots I-1, l = 0 \ldots L-1) \end{aligned}\right\} . \tag{11}$$

In this case, the simple (albeit again perhaps not the most efficient) expedient could be to compute $(w_{0,0}, z_{0,0})$, $(w_{0,L-1}, z_{0,L-1})$ and $(w_{I-1,0}, z_{I-1,0})$ rigorously, and then use (11) to compute $(\Delta w, \Delta z, c_w, v_z)$,, see Fig. 3.

Note that this procedure uses only three out of the four "corners" of a window, thus introducing a kind of asymmetry in the treatment of the samples. The obvious way out would be to also compute (w_{I-1}, z_{L-1}) and to do a full 2-d linear interpolation. This is not necessary in a first implementation of the LPC concept, but could be an item for improvement later on. The possible errors (introduced by the nonlinearity of the scan

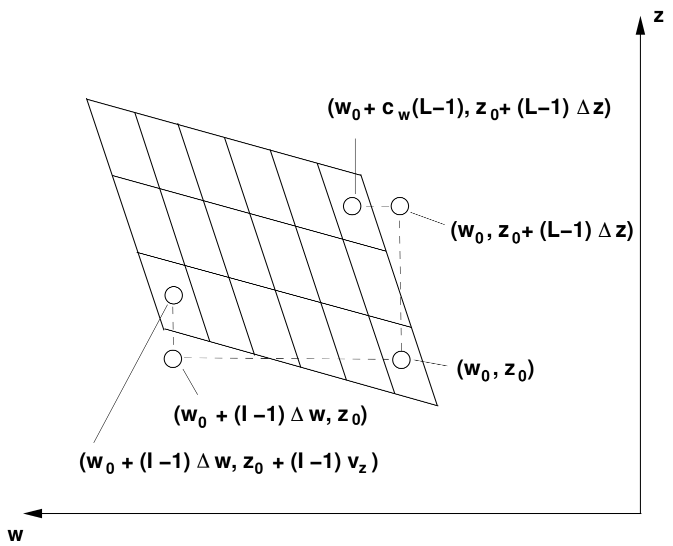

FIGURE 3: Local scan coordinates (w, z) for individual samples, with and without shear terms.

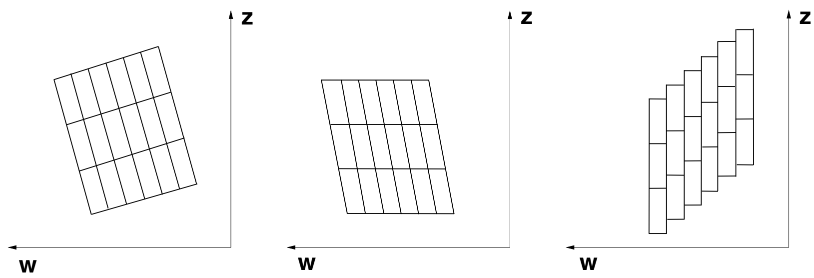

FIGURE 4: The underlying causes of the shear terms illustrated; see text.

motion and the calibration function over the small range given by the size of a window) should be small, and for a given source they should not be systematic. On the other hand, using the fourth "corner" of the window might be used as a convenient way to create a unit test for the "3-corner" version.

It might be instructive to briefly consider the causes of the shear terms and their probable sizes. This subject is illustrated in Fig.. 4. This will be done in the rest of the present subsection. It may be skipped without disadvantages for the implementation or usage of the local plane coordinates.

The left-hand panel shows that shear terms may arise in the w, z coordinate system even if the pixels/samples form an exactly rectangular pattern on the sky. The reason is a possible rotation of the image of the CCDs on the sky with respect to the field-of-view reference system (the latter being defined by the great circle joining the two projections of some 'central' FPA point on the sky, in this way setting the (η, ζ) field coordinates, and in consequence the (w, z) local scan coordinates). Due to the TDI operation of the CCDs, such a rotation leads to a c_w term only, but not to a v_z term (see below). Possible causes of such a rotation are at least threefold: the imprecision of the glueing of the CCDs onto the focal-plane array (FPA), a rotation of the FPA with respect to the above-mentioned great circle (the 'image rotation' known from Hipparcos), and an across-scan offset of the 'central' FPA point from that great circle (the 'differential image rotation' known from Hipparcos). Appendix A shows that the effects are probably significant.

The center panel shows an actual shear of the CCD image on the sky. It may be created by optical distortion. Again, due to the TDI operation of the CCDs, this effect leads to a c_w term only, but not to a v_z term (see below).

The right-hand panel shows the effect of an across-scan attitude motion. This time the samples/pixels shown in the panel do not indicate the projection of the physical CCD samples/pixels on the sky, but the projection of TDI samples, i.e. of the actual Gaia data items. The location of a TDI sample represents the average location on the sky of the corresponding charge cloud over the actual exposure time for this sample. An across-scan attitude motion slowly shifts the location of a given CCD column in the across-scan direction on the sky, and thus the across-scan position of the TDI samples derived from that column. Across-scan attitude motion is the *only* effect that creates a v_z term.

As an aside we would like to make a remark on the 'sample dimensions' $(\Delta w, \Delta z)$. Due to the TDI operation of the CCDs, the along-scan sample dimension (i.e. Δw) has no connection whatsoever with the physical size of the samples/pixels on the sky. Thus it does not change by e.g. focal-length variations, optical distortion or other optical effects. Instead, it is determined solely by the ratio between the along-scan attitude motion and the TDI time interval. It strictly measures the angle by which the spacecraft has rotated between two successive TDI clock strokes, averaged of over the actual exposure time for any given star image. In stark contrast, the across-scan sample dimension (i.e. Δz) is solely determined by optical effects and by the physical size of the CCD pixels, i.e. by the focal length, optical distortion, the FPA temperature and so on. It strictly measures the angular offset between neigbouring CCD columns, averaged over the path that the star image has taken on the CCD until being read out at the trailing edge of the CCD.

4.4 Combining elementary and window data

Several data items appear both in Table 2 and 3, and it might be desirable to combine the two data structures (Table 4). Since most of the overlap is in the headers, the saving is moderate in terms of the total data volume.

5 Planetary objects

The concept of local plane coordinates as presented in this document cannot be used for solar-system objects, for at least two fundamental reasons:

1. There is no obvious choice for the reference point (α_0, δ_0). In fact, since most of the planetary objects over the course of five years move across the whole sky, no possible choice of (α_0, δ_0) for a given object could be used throughout the whole Gaia mission.

2. For a planetary object it is impossible to compute a coordinate direction without precisely knowing the orbit. Both the (huge) parallactic effect as well as the relativistic light bending strongly depend on the distance to the object.

Therefore it seems unavoidable that the planetary-objects task has to make use of the original Gaia centroids, corresponding to proper directions. Further thinking might lead to an interface that avoids explicit use of the Gaia calibration and attitude files, but the Gaia orbit and solar-system ephemeris are fundamentally unavoidable. It is thus doubtful whether a special astrometric interface would be worthwhile in the case of planetary objects.

As an aside we mention that in the case of planetary objects any averaging over several CCD transits should not be done. The angular motions of the objects are quite large, and also the changes of the actual scan direction over a minute of time are relevant.

6 Discussion

Since the appearance of issue 2 of GAIA-LL-061 it has been decided that LPCs will be used by DPAC, and that indeed (as indicated above) they will be produced by CU4 and CU5 internally, i.e. “on the fly”. Nevertheless we keep the following discussion from issue 2, to illuminate the background of this decision.

The decision includes the agreement that the necessary software module will be provided by CU3. The necessary input data (source catalogue, attitude, calibration, orbit and ephemerides) will be available at the CU4 and CU5 processing centres by virtue of the regular deliveries of the Main Data Base.

Using local coordinates as described above for the astrometric data input to the object processing, either at the elementary or window level, gives two very significant advantages:

1. It relieves the object processing and 2-d imaging workpackages from the tedious, complicated and potentially error-prone mechanics of making all the transformations and corrections involving the attitude, calibration and orbital data.
2. It provides an extremely simple and transparent data interface, where all the relevant information for a specific object may be collected in a single, small file.

Possible disadvantages are:

1. More processing is required to generate the data in local coordinates than just to copy the relevant files (elementary or raw data, plus attitude, etc). Actually, the total processing may not be much affected, since the processing is merely shifted from one place to another.
2. The data volumes to transfer may increase, since the local scan coordinates require additional information to be provided along with the data – position angle of the scan, pixel dimensions in local scan coordinates, etc. See below.
3. Definitely, similar data need to be transferred several times as the attitude and calibration data are modified in the global AGIS iterations. Even the original (pixel-coordinate) centroids and the sample values will change, although less frequently.

Concerning item 2 we have to compare the data sizes in Tables 2–4 with the typical sizes of the astrometric elementary and window data per object. For the astrometric elementaries we need the time (long) and transverse pixel coordinate (int) per CCD transit, the CCD row (int) and flags (int) per FOV transit, and an identifier (long) per object; the total size (assuming $N = 80$, $M = 10$) is 11 kBytes per object. This is nearly 3 times the amount in Table 2, and about a quarter of that in Table 4. For the sample data we need in addition 6 sample values (float) per CCD transit (assuming $IL = 6$), increasing the size to 32 kByte per object. The saving compared to Table 4 is only 30% or 14 kByte per object. This can be further reduced if the data for the local plane coordinates are directly combined with the astrometric elementaries.

Note however that the attitude file for the whole mission (5 years) will be about 1 GByte, and the geometric calibration files will be at least of the same order. These need to be transferred in addition to the elementary/sample data in case the local scan coordinates are not used. Thus, whether or not there is a net saving in the total data volume depends on the total number of objects that will be treated by the object processing and the 2-d imaging, and whether the transfer of the relevant data for a subset of the whole 10^9 Gaia objects will be practical. The break-even number is of order (3 GByte)/(6 kByte) = 500 000. Since it is likely that many millions of objects will be treated in the object processing task, and essentially all objects in the 2-d imaging task, the raw sample data plus attitude etc. would still give the smaller volume, but only by about 15%. The penalty would be to install a lot of additional astrometric and data handling software on the side of the object processing and 2-d imaging.

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Appendix A: The need of AC field coordinates to compute AL local coordinates

The practical computation of local scan coordinates (w, z) from measured centroids (t, n, m) was briefly sketched in Section 4.1, where t is the observed transit time (also called along-scan pixel coordinate), n is the CCD number and m is the across-scan pixel coordinate. In slightly more detail this computation runs as follows:

Step 1: $(t, n, m) \rightarrow (t, \eta, \zeta)$, field coordinates, using the geometric calibration
Step 2: $(t, \eta, \zeta) \rightarrow \boldsymbol{u}$, proper direction, using the attitude
Step 3: $(t, \boldsymbol{u}) \rightarrow (t, \boldsymbol{c})$, coordinate direction, using ephemerides (sun and spacecraft)
Step 4: $(t, \boldsymbol{c}) \rightarrow (t, a, d)$, local celestial coordinates, using reference point (α_0, δ_0)
Step 5: $(t, a, d) \rightarrow (t, w, z)$, local scan coordinates, using attitude information.

Generally, in the AF the coordinate m, and thus ζ is not directly known from the measurements. Therefore the above transformation steps could not be done to the necessary high precision unless these coordinates would be derived from external information. Such information can either be the measured AC coordinate of the same celestial object from the (always available) immediately preceding SM transit, extrapolated to the time of the AF transit under consideration. Alternatively it could be an approximate knowledge of the object's position from a-priori astrometric parameters (position, proper motion, parallax).

This appendix briefly looks at the question which precision the external AC information needs to have in order not to disturb the inherent precision of the AL measurements in the AF. Integer-pixel precision is not sufficient, as will be seen immediately.

Let the glueing of the CCDs onto the Gaia focal-plane array be precise to $10\,\mu$m, for the physical location of any corner of a CCD. This implies a typical rotation of the CCDs (with respect to their nominal orientation) of $10\,\mu\text{m}/60\,\text{mm} = 10\,\mu\text{as}/60\,\text{mas} = 30\,\mu\text{as}$ per AC pixel. Thus, if the AC coordinate would be known to 1 AC pixel only, AL transformation errors of several dozen microarcsec would result. Effects of similar size must be expected from optical distortion, image rotation etc. (a different assumption for the glueing precision would thus not help much).

An AF-external knowledge of the AC position of all objects to about 1 mas is thus desirable in order to ensure correctness of the transformation to the order of $1\,\mu$as. It is clear that this will easily be achieved in the later stages of the Gaia mission and data reduction. For the majority of the faint stars it will not be available in the early stages, neither from the SM transits, nor from a star catalogue. However, even for faint stars the SM transits will be sufficient from the very beginning, since the precision of their individual AF measurements is lower, too.

Appendix B: Calculation of the scan direction θ

The $+w$ axis in Fig. 1 points in the direction of the unit vector $(\boldsymbol{z} \times \boldsymbol{r}_0)/C$, where $C = |\boldsymbol{z} \times \boldsymbol{r}_0|$ is a normalizing factor, and $\boldsymbol{z}$ is the z axis (the third axis) of the Scanning Refererence System (SRS), as defined in [1]. Recalling that the $+d$ axis points in the direction of unit vector $\boldsymbol{q}_0$ we have

$$C \cos\theta = (\boldsymbol{z} \times \boldsymbol{r}_0)'\boldsymbol{q}_0 = (\boldsymbol{r}_0 \times \boldsymbol{q}_0)'\boldsymbol{z} = -\boldsymbol{p}_0'\boldsymbol{z} \tag{12}$$

and since $+a$ points in the direction of unit vector $\boldsymbol{p}_0$ we similarly have

$$C \sin\theta = (\boldsymbol{z} \times \boldsymbol{r}_0)'\boldsymbol{p}_0 = (\boldsymbol{r}_0 \times \boldsymbol{p}_0)'\boldsymbol{z} = \boldsymbol{q}_0'\boldsymbol{z} \tag{13}$$

The position angle can therefore be computed as

$$\theta = \mathrm{atan2}(\boldsymbol{q}_0'\boldsymbol{z}, -\boldsymbol{p}_0'\boldsymbol{z}) \tag{14}$$

The unit vector $\boldsymbol{z}$ is directly obtained from the attitude quaternion at the time of the transit.

Appendix C: Computation of epoch astrometry from actual observations

This Appendix summarises the formulae (or code snippets) needed to compute the epoch astrometry (i.e. the LPC centroids and ancillary data) from actual observations.

For a given source, a reference epoch T and a reference point (α_0, δ_0) must be chosen. In practice these are the reference epoch of the AGIS solution and the AGIS position of the source at the reference epoch. The reference point defines the reference triad as in Eq. (1), i.e.

$$\boldsymbol{p}_0 = \begin{bmatrix} -\sin\alpha_0 \\ \cos\alpha_0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{q}_0 = \begin{bmatrix} -\sin\delta_0\cos\alpha_0 \\ -\sin\delta_0\sin\alpha_0 \\ \cos\delta_0 \end{bmatrix}, \qquad \boldsymbol{r}_0 = \begin{bmatrix} \cos\delta_0\cos\alpha_0 \\ \cos\delta_0\sin\alpha_0 \\ \sin\delta_0 \end{bmatrix}. \tag{15}$$

Importantly, these vectors are fixed for a given source.[7]

For each CCD observation, the following input data are then needed:

1. the observation time t_obs expressed in TCB;

2. the observed coordinate direction $\boldsymbol{c}$ to the source at time t_obs. This could be obtained from the observation by a sequence of transformations as in

```
WrsEvent observation = ...
BcrsEvent bcrsEvent = observation.toSrs().toComrs().toBcrs();
GVector3d c = bcrsEvent.getDirection();
```

   Effectively, this removes aberration and gravitational light deflection (but not the parallax) from the observed (proper) direction;

3. the spin axis $\boldsymbol{z}$ of the SRS at time t_obs, expressed in the CoMRS. This could be obtained from the attitude quaternion $\mathbf{q}$ at time t_obs as

```
Quaternion q = ...
GVector3d z = new GVector3d(0., 0., 1.);
z.rotateVectorByQuaternion(q);
```

4. the barycentric position $\boldsymbol{b}_\mathrm{G}$ and velocity $\boldsymbol{v}_\mathrm{G}$ of Gaia at time t_obs, expressed in m and m s^{-1}.

The position angle of the scan is obtained as in Eq. (14), i.e.

$$\theta = \mathrm{atan2}(\boldsymbol{q}_0'\boldsymbol{z}, -\boldsymbol{p}_0'\boldsymbol{z}). \tag{16}$$

[7]For a resolved binary, where the components have different AGIS solutions, it simplifies the analysis if the epoch astrometry of both components is expressed with respect to the same reference point, for example half-way between the components at the reference epoch. This requires that the epoch astrometry is transformed from one reference point to another, which can be done as detailed in Appendix D.

The parallax factors in a and d, and the barycentric correction Δt are obtained as[8]

$$f_a = -\boldsymbol{p}_0'\boldsymbol{b}_\mathrm{G}/A\,, \quad f_d = -\boldsymbol{q}_0'\boldsymbol{b}_\mathrm{G}/A\,, \quad \Delta t = \boldsymbol{r}_0'\boldsymbol{b}_\mathrm{G}/c\,, \tag{17}$$

where A is the astronomical unit in m (`Nature:AstronomicalUnit_Meter`) and c is the speed of light (`Nature:VelocityOfLight_Constant_Vacuum`). From this, the parallax factors in w and z are obtained as in Eq. (9), i.e.

$$\left.\begin{aligned} f_w &= f_a\sin\theta + f_d\cos\theta \\ f_z &= -f_a\cos\theta + f_d\sin\theta \end{aligned}\right\}\,, \tag{18}$$

and the barycentric epoch of observation is as in Eq. (5), i.e.

$$t_\mathrm{B} = t_\mathrm{obs} + \Delta t\,. \tag{19}$$

Finally the offsets in α and δ, obtained by Eq. (2),

$$a = \frac{\boldsymbol{p}_0'\boldsymbol{c}}{\boldsymbol{r}_0'\boldsymbol{c}}\,, \qquad d = \frac{\boldsymbol{q}_0'\boldsymbol{c}}{\boldsymbol{r}_0'\boldsymbol{c}}\,, \tag{20}$$

are converted to offsets AL and AC by means of Eq. (6), i.e.

$$\left.\begin{aligned} w &= a\sin\theta + d\cos\theta \\ z &= -a\cos\theta + d\sin\theta \end{aligned}\right\}\,. \tag{21}$$

For transforming the uncertainties σ_η, σ_ζ in the AL and AC field angles, we also need the (approximate) AC field angle and the (classical) aberration factor discussed in Sect. 2. The AC field angle ζ is obtained to sufficient accuracy from

$$\sin\zeta = \boldsymbol{z}'\boldsymbol{r}_0\,. \tag{22}$$

Thus ζ is actually the field angle of the reference point (cf. Appendix D). For the aberration effect we have

$$\gamma = \boldsymbol{r}_0'\boldsymbol{v}_\mathrm{G}/c\,, \tag{23}$$

which is in the range $|\gamma| \lesssim 10^{-4}$. Then

$$\sigma_w = (1+\gamma)\sigma_\eta\cos\zeta\,, \quad \sigma_z = (1+\gamma)\sigma_\zeta\,. \tag{24}$$

The aberration factor $1+\gamma$ corrects for the apparent contraction of images (and hence of the error ellipses) in the direction towards which the observer moves. (The much smaller effect of the gravitational deflection is here neglected.) Because $|\zeta| \lesssim 0.4^\circ$ it follows that $1-\cos\zeta \lesssim 2.4\times 10^{-5}$, so the aberration effect is more important than the $\cos\zeta$ effect.

The file `lpcBS.fits` contains test data for the epoch astrometry. They were generated from an AGIS solution for Barnard's star (`source_id` = 4472832822427129472), using $T = 2017.5$, $\alpha_0 = 269.4481674047229^\circ$, $\delta_0 = +4.743738338140268^\circ$.

[8] We adopt here the convention that the correction is positive when the event at the barycentre is later than the time of observation at Gaia. The name 'Roemer delay' is sometimes used for the same quantity but with the opposite sign.

TOPCAT(96): Table Columns

Table Columns for 96: lpcBS.fits

Δ	Index	Visible	Name	$ID	Class	Description	Datatype
0		☐	Index	$0	Long	Table row index	
1	1	☑	tObs_rev	$1	Double	time of observation as OBMT [rev]	double
2	2	☑	cX	$2	Double	ICRS X component of coordinate direction	double
3	3	☑	cY	$3	Double	ICRS Y component of coordinate direction	double
4	4	☑	cZ	$4	Double	ICRS Z component of coordinate direction	double
5	5	☑	zeta	$5	Double	AC field angle [rad]	double
6	6	☑	qX	$6	Double	quaternion X component	double
7	7	☑	qY	$7	Double	quaternion Y component	double
8	8	☑	qZ	$8	Double	quaternion Z component	double
9	9	☑	qW	$9	Double	quaternion W component	double
10	10	☑	bXG	$10	Double	BCRS X component of Gaia's barycentric position [m]	double
11	11	☑	bYG	$11	Double	BCRS Y component of Gaia's barycentric position [m]	double
12	12	☑	bZG	$12	Double	BCRS Z component of Gaia's barycentric position [m]	double
13	13	☑	vXG	$13	Double	BCRS X component of Gaia's barycentric velocity [m/s]	double
14	14	☑	vYG	$14	Double	BCRS Y component of Gaia's barycentric velocity [m/s]	double
15	15	☑	vZG	$15	Double	BCRS Z component of Gaia's barycentric velocity [m/s]	double
16	16	☑	tB_rev	$16	Double	barycentric time of observation as OBMT [rev]	double
17	17	☑	a	$17	Double	LPC coordinate in RA [rad]	double
18	18	☑	d	$18	Double	LPC coordinate in Dec [rad]	double
19	19	☑	w	$19	Double	LPC coordinate AL [mas]	double
20	20	☑	z	$20	Double	LPC coordinate AC [mas]	double
21	21	☑	theta	$21	Double	position angle of scan [rad]	double
22	22	☑	fw	$22	Double	parallax factor AL	double
23	23	☑	fz	$23	Double	parallax factor AC	double
24	24	☑	aberr	$24	Double	aberration	double
25	25	☑	DeltaT	$25	Double	barycentric correction, tB-tObs [s]	double

FIGURE 5: Column descriptions for the test data file `lpcBS.fits`.

Appendix D: Changing the reference point

Occasionally it is desirable to change the reference point (α_0, δ_0) for a set of epoch astrometry to some other point (α_1, δ_1) very nearby on the sky (within an arcmin or so; cf. Sect. 2). A typical example could be a resolved binary, where the components have separate sets of epoch astrometry, but where the user wants to make a joint solution for the system. It is then necessary to transform the epoch astrometry of one component to the reference point of the other component, or to transform both sets to a third reference point. This appendix briefly explans how this can be done. Not only the epoch positions (w, z) need to be transformed, but also the ancillary data, in particular θ. To distinguish the two data sets we use subscript 0 for the original data referring to (α_0, δ_0), as in w_0 and θ_0, and subscript 1 for the transformed data referring to (α_1, δ_1), as in w_1 and θ_1.

The new reference triad $[\,\boldsymbol{p}_1\ \boldsymbol{q}_1\ \boldsymbol{r}_1\,]$ is immediately obtained in analogy with Eq. (15).

The observation times $t_{\rm obs}$ and are of course unchanged by the transformation. The same is in principle be the case for the barycentric epochs $t_{\rm B}$, but if the barycentric correction Δt is computed according to Eq. (17), this is not strictly true. If the change in reference

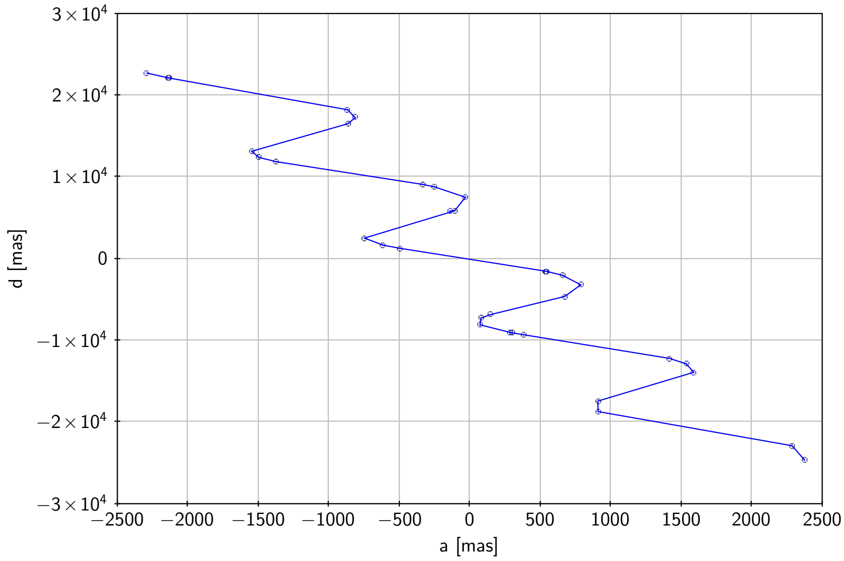

FIGURE 6: A plot of the (a, d) coordinates for the test data in `lpcBS.fits`.

position is less than 1 arcmin, we have $|\boldsymbol{r}_1 - \boldsymbol{r}_0| < 3 \times 10^{-4}$ and $|\Delta t_1 - \Delta t_0| < 0.15$ s, which has a negligible impact on the calculations even in the most extreme cases of high proper motions. We can therefore disregard this small discrepancy and transform Δt consistently with the parallax factors. This guarantees that the transformation is strictly reversible.

For a given epoch position (w_0, z_0) we have from Eq. (7) the offset coordinates relative to the original reference point,

$$\left.\begin{aligned} a_0 &= w_0 \sin\theta_0 - z_0 \cos\theta_0 \\ d_0 &= w_0 \cos\theta_0 + z_0 \sin\theta_0 \end{aligned}\right\} , \tag{25}$$

and hence from Eq. (3) the coordinate direction of the observation,

$$\boldsymbol{c} = \frac{\boldsymbol{r}_0 + \boldsymbol{p}_0 a_0 + \boldsymbol{q}_0 d_0}{(1 + a_0^2 + d_0^2)^{1/2}} . \tag{26}$$

The coordinate direction is of course independent of the choice of reference point (hence it has no subscript here), which immediately allows us to compute the new offset coordinates in analogy with Eq. (2),

$$a_1 = \frac{\boldsymbol{p}_1' \boldsymbol{c}}{\boldsymbol{r}_1' \boldsymbol{c}} , \qquad d_1 = \frac{\boldsymbol{q}_1' \boldsymbol{c}}{\boldsymbol{r}_1' \boldsymbol{c}} . \tag{27}$$

It can be noted that the result of Eq. (27) is invariant to an arbitrary scaling of the vector $\boldsymbol{c}$; thus, if $\boldsymbol{c}$ is only used for this calculation one can omit the normalisation in Eq. (26).

To make the transformation to AL and AC we now need θ_1, the position angle of the scan at the new reference point. This can be obtained from Eq. (16), provided that the SRS $\boldsymbol{z}$ axis is known. Its direction in the reference triad $[\,\boldsymbol{p}_0\;\boldsymbol{q}_0\;\boldsymbol{r}_0\,]$ is completely specified by θ_0 and ζ_0, the AC field angle of the reference point at the time of observation. Specifically, we have

$$\boldsymbol{z} = -\boldsymbol{p}_0 \cos\zeta_0 \cos\theta_0 + \boldsymbol{q}_0 \cos\zeta_0 \sin\theta_0 + \boldsymbol{r}_0 \sin\zeta_0\,. \tag{28}$$

We then have in analogy with Eq. (16)

$$\theta_1 = \mathrm{atan2}(\boldsymbol{q}_1'\boldsymbol{z}, -\boldsymbol{p}_1'\boldsymbol{z})\,, \tag{29}$$

from which

$$\left.\begin{aligned} w_1 &= a_1 \sin\theta_1 + d_1 \cos\theta_1 \\ z_1 &= -a_1 \cos\theta_1 + d_1 \sin\theta_1 \end{aligned}\right\}\,. \tag{30}$$

To compute the new parallax factors in analogy with Eq. (17) we need the barycentric position of Gaia, $\boldsymbol{b}_\mathrm{G}$. From Eq. (17) we have

$$\boldsymbol{b}_\mathrm{G}/A = -\boldsymbol{p}_0 f_{a0} - \boldsymbol{q}_0 f_{d0} + \boldsymbol{r}_0 \Delta t_0/\tau_\mathrm{A}\,, \tag{31}$$

where

$$\left.\begin{aligned} f_{a0} &= f_{w0}\sin\theta_0 - f_{z0}\cos\theta_0 \\ f_{d0} &= f_{w0}\cos\theta_0 + f_{z0}\sin\theta_0 \end{aligned}\right\} \tag{32}$$

are the parallax factors in (a_0, d_0) and $\tau_\mathrm{A} = A/c = 499.004783836156$ s is the astronomical unit in light-seconds. The transformed parallax factors in a_1, d_1, and the barycentric correction, are now obtained in analogy with Eq. (17), i.e.

$$f_{a1} = -\boldsymbol{p}_1'(\boldsymbol{b}_\mathrm{G}/A)\,, \quad f_{d1} = -\boldsymbol{q}_1'(\boldsymbol{b}_\mathrm{G}/A)\,, \quad \Delta t_1 = \boldsymbol{r}_1'(\boldsymbol{b}_\mathrm{G}/A)\tau_A\,, \tag{33}$$

from which

$$\left.\begin{aligned} f_{w1} &= f_{a1}\sin\theta_1 + f_{d1}\cos\theta_1 \\ f_{z1} &= -f_{a1}\cos\theta_1 + f_{d1}\sin\theta_1 \end{aligned}\right\}\,, \tag{34}$$

Finally,

$$\sin\zeta_1 = \boldsymbol{z}'\boldsymbol{r}_1\,. \tag{35}$$

The transformation of w, z, ζ, θ, f_w, f_z, and Δt from one reference point to another is strictly reversible: when transformed back to the original reference point, the original values are recovered up to numerical rounding errors (Table 1).

Table 1: A numerical example demonstrating the reversibility of the transformation in Appendix D from one reference point to another. The example is the first observation of Barnard's star in the file `lpcBS.fits`. In the column 'Transformed value' the reference point has been increased by 0.001 rad in each coordinate, i.e. almost 5 arcmin in total. The last column shows that a back-transformation to the original reference point recovers the original (w, z) and ancillary data to the full numerical precision, or about 10^{-8} mas.

Quantity	Unit	Original value	Transformed value	Back-transformed value
α_0	rad	4.7027576846772785	4.7037576846772788	4.7027576846772785
δ_0	rad	+0.0827938528536318	+0.0837938528536318	+0.0827938528536318
ζ	rad	+0.0037167717231132	+0.0035033228528330	+0.0037167717231132
θ	rad	−2.2061999999999999	−2.2061117644403287	−2.2061999999999999
w	mas	+12705.438829000001	+300555.178744405974	+12705.438828997365
z	mas	+21227.942417999999	+65255.769314761019	+21227.942417996732
f_w	–	−0.5642410000000000	−0.5636542974445808	−0.5642410000000000
f_z	–	+0.7025840000000000	+0.7026714371690873	+0.7025839999999999
Δt	s	−210.89541299999999	−211.21329584692054	−210.89541299999985

Appendix E: An example using the epoch astrometry

In this Appendix we fit the astrometric parameters to the epoch astrometry for Barnard's star, as given in the file `lpcBS.fits`. We begin by recalling the "standard model" of stellar kinematics, as defined for example by Eqs. (4)–(6) in [5]; this defines the precise meaning of the astrometric parameters and allows us to calculate the epoch astrometry as a function of time for a given set of parameters, as well as the partial derivatives of the observables with respect to the parameters. The actual model fitting is then a straight-forward application of the (slightly non-linear) least-squares method.

E.1. The standard model

The standard model of stellar kinematics assumes constant space velocity relative to the Solar System Barycentre (SSB), but ignores light-time effects beyond the Solar System.[9] The motion is modelled in the Barycentric Celestial Reference System (BCRS), with TCB as the time coordinate. The coordinate direction from Gaia to the star at the time of observation $t_{\rm obs}$ is

$$\boldsymbol{c}(t_{\rm obs}) = \langle \boldsymbol{b}_T + (t_{\rm B} - T)\boldsymbol{v} - \boldsymbol{b}_{\rm G}(t_{\rm obs})\rangle \ , \tag{36}$$

where the angular brackets $\langle\,\rangle$ signify vector normalisation, $\langle\boldsymbol{x}\rangle = \boldsymbol{x}|\boldsymbol{x}|^{-1}$. T is the reference epoch for the astrometry, $\boldsymbol{b}_T$ the barycentric vector to the star at the reference epoch, $\boldsymbol{v}$ the space velocity of the star, $\boldsymbol{b}_{\rm G}(t)$ the Gaia ephemeris, and $t_{\rm B} = t + \boldsymbol{c}'\boldsymbol{b}_{\rm G}(t_{\rm obs})c^{-1}$ the barycentric time of the observation. From here on, we suppress the argument $t_{\rm obs}$ to $\boldsymbol{c}$ and $\boldsymbol{b}_{\rm G}$, and introduce $\tau = t_{\rm B} - T$ for brevity.

Given the arbitrary[10] reference position (α_0, δ_0), we can express the three vectors $\boldsymbol{b}_T$, $\boldsymbol{v}$, and $\boldsymbol{b}_{\rm G}$ in terms of the reference triad $[\,\boldsymbol{p}_0\ \boldsymbol{q}_0\ \boldsymbol{r}_0\,]$ as

$$\left.\begin{aligned}
\boldsymbol{b}_T &= \boldsymbol{p}_0(\boldsymbol{p_0}'\boldsymbol{b}_T) + \boldsymbol{q}_0(\boldsymbol{q_0}'\boldsymbol{b}_T) + \boldsymbol{r}_0(\boldsymbol{r_0}'\boldsymbol{b}_T)\,,\\
\boldsymbol{v} &= \boldsymbol{p}_0(\boldsymbol{p_0}'\boldsymbol{v}) + \boldsymbol{q}_0(\boldsymbol{q_0}'\boldsymbol{v}) + \boldsymbol{r}_0(\boldsymbol{r_0}'\boldsymbol{v})\,,\\
\boldsymbol{b}_{\rm G} &= \boldsymbol{p}_0(\boldsymbol{p_0}'\boldsymbol{b}_{\rm G}) + \boldsymbol{q}_0(\boldsymbol{q_0}'\boldsymbol{b}_{\rm G}) + \boldsymbol{r}_0(\boldsymbol{r_0}'\boldsymbol{b}_{\rm G})\,,
\end{aligned}\right\} \tag{37}$$

where the bracketed expressions are the projections of the vectors on the triad. Because of the normalisation operator in Eq. (36) we can divide each of the equations in (37) by any positive number and still get the same $\boldsymbol{c}$. We choose to divide by $b_0 = \boldsymbol{r}_0'\boldsymbol{b}_T$, that is

[9] Ignoring the (large and uncertain) light time $|\boldsymbol{b}_T|c^{-1}$ from the star to the observer is normal practice in astrometry. Over time intervals of a few decades this has negligible impact for the *modelling* of the observations, but possibly not for their *interpretation* in terms of physical motions in our Galaxy. See [6] for further discussion.

[10] "Arbitrary" in the sense that the reference position, at least in principle, does not have to be anywhere near the actual position of the star at any time, as long as $\boldsymbol{r}_0'\boldsymbol{b}_T > 0$. In practice, the reference point must be reasonably close to the observations for the subsequent definitions of parallax etc to make sense.

the barycentric distance to the star at the reference epoch projected in the direction of the reference point.

At epoch T and relative to the reference point (α_0, δ_0), the astrometric parameters are now defined as

$$\left.\begin{array}{r}a_0 = \boldsymbol{p}_0'\boldsymbol{b}_T/b_0\,, \quad d_0 = \boldsymbol{q}_0'\boldsymbol{b}_T/b_0\,, \quad \varpi_0 = A/b_0\,, \\ \mu_{\alpha*0} = \boldsymbol{p}_0'\boldsymbol{v}/b_0\,, \quad \mu_{\delta 0} = \boldsymbol{q}_0'\boldsymbol{v}/b_0\,, \quad \mu_{r0} = \boldsymbol{r}_0'\boldsymbol{v}/b_0\,. \end{array}\right\} \tag{38}$$

It is seen that $\boldsymbol{c}$ is obtained by normalising the almost-unit vector

$$\tilde{\boldsymbol{c}} = \boldsymbol{p}_0(a_0 + \tau\mu_{\alpha*0} + f_a\varpi_0) + \boldsymbol{q}_0(a_0 + \tau\mu_{\delta 0} + f_d\varpi_0) + \boldsymbol{r}_0(1 + \tau\mu_{r0} + f_r\varpi_0)\,, \tag{39}$$

where

$$f_a = -\boldsymbol{p}_0'\boldsymbol{b}_\mathrm{G}/A\,, \quad f_d = -\boldsymbol{q}_0'\boldsymbol{b}_\mathrm{G}/A\,, \quad f_r = -\boldsymbol{r}_0'\boldsymbol{b}_\mathrm{G}/A\,. \tag{40}$$

Equation (2) then gives

$$a = \frac{\boldsymbol{p}_0'\tilde{\boldsymbol{c}}}{\boldsymbol{r}_0'\tilde{\boldsymbol{c}}} = \frac{a_0 + \tau\mu_{\alpha*0} + f_a\varpi_0}{1 + \tau\mu_{r0} + f_r\varpi_0}\,, \qquad d = \frac{\boldsymbol{q}_0'\tilde{\boldsymbol{c}}}{\boldsymbol{r}_0'\tilde{\boldsymbol{c}}} = \frac{d_0 + \tau\mu_{\delta 0} + f_d\varpi_0}{1 + \tau\mu_{r0} + f_r\varpi_0}\,, \tag{41}$$

from which (w, z) are obtained by means of Eq. (6).

In the above expressions some care must be exercised concerning the angular units, unless radians are used throughout. It is convenient to use mas (or mas yr^{-1}) for a, d, w, z, as well as for the astrometric parameters a_0, ..., μ_{r0}, while τ is expressed in (Julian) years and f_a, f_d, and f_r are dimensionless. In the denominator of Eq. (41) it is however necessary to have $\tau\mu_{r0}$ and $f_r\varpi_0$ in radians; to avoid any ambiguity, we introduce hereafter the explicit conversion factor u when needed (see below).

As a side remark we note that the astrometric parameters in Eq. (38) are defined relative to the reference position (α_0, δ_0); in particular a_0 and d_0 specify a small, but in general non-zero, offset of the coordinate direction from the reference point at epoch T. This "differential form" is very useful for fitting the kinematic model to the observations, because it is very nearly linear in all six parameters. However, it is not the most convenient way to specify the astrometric parameters, for example in a catalogue. The reason is that there are eight quantities to specify (namely, the six astrometric parameters, plus the reference position), although the model only has six degrees of freedom. The redundancy can however be removed by selecting the reference position such that $a_0 = d_0 = 0$, and regarding the reference position α_0, δ_0 as the first two astrometric parameters instead of a_0 and d_0. Indeed, this is how the astrometric parameters in the Gaia Archive must be interpreted: for example, the proper motion components are measured along the vectors $\boldsymbol{p}_0$ and $\boldsymbol{q}_0$ defined by the reference position given by the first two astrometric parameters. This has a subtle effect for the propagation of the uncertainties in proper motion (see p. 96 in [2]).

To summarise, we have

$$\left.\begin{aligned} w &= \frac{(a_0 + \tau\mu_{\alpha*0})\sin\theta + (d_0 + \tau\mu_{\delta0})\cos\theta + f_w\varpi_0}{D}\,, \\ z &= \frac{-(a_0 + \tau\mu_{\alpha*0})\cos\theta + (d_0 + \tau\mu_{\delta0})\sin\theta + f_z\varpi_0}{D}\,, \end{aligned}\right\} \tag{42}$$

where

$$D = 1 + (\tau\mu_{r0} + f_r\varpi_0)u\,, \tag{43}$$

and $u = \pi/(180 \times 60 \times 60 \times 1000)$ if mas are used. Equations (42) and (43) provide the complete model of the observed quantities (w, z) in terms of the six astrometric parameters a_0, d_0, ϖ_0, $\mu_{\alpha*0}$, $\mu_{\delta0}$, and $\mu_{r0} = v_{r0}\varpi_0/A$. The remaining quantities $\tau = t_\mathrm{B} - T$, f_a, f_d, f_r, and θ are assumed to be known with zero uncertainty.

The partial derivatives of the observables w and z with respect to the six astrometric parameters are readily obtained from the previous expressions; they are:

$$\left.\begin{aligned} \frac{\partial w}{\partial a_0} &= \frac{\sin\theta}{D}\,, & \frac{\partial z}{\partial a_0} &= -\frac{\cos\theta}{D}\,, \\ \frac{\partial w}{\partial d_0} &= \frac{\cos\theta}{D}\,, & \frac{\partial z}{\partial d_0} &= \frac{\sin\theta}{D}\,, \\ \frac{\partial w}{\partial \varpi_0} &= \frac{f_w - wf_ru}{D}\,, & \frac{\partial z}{\partial \varpi_0} &= \frac{f_z - zf_ru}{D}\,, \\ \frac{\partial w}{\partial \mu_{\alpha*0}} &= \frac{\tau\sin\theta}{D}\,, & \frac{\partial z}{\partial \mu_{\alpha*0}} &= -\frac{\tau\cos\theta}{D}\,, \\ \frac{\partial w}{\partial \mu_{\delta0}} &= \frac{\tau\cos\theta}{D}\,, & \frac{\partial z}{\partial \mu_{\delta0}} &= \frac{\tau\sin\theta}{D}\,, \\ \frac{\partial w}{\partial \mu_{r0}} &= -\frac{w\tau u}{D}\,, & \frac{\partial z}{\partial \mu_{r0}} &= -\frac{z\tau u}{D}\,. \end{aligned}\right\} \tag{44}$$

E.2. Fitting the standard model to the epoch astrometry for Barnard's star

The epoch astrometry in the file `lpcBS.fits` refer to the reference epoch $T = 2017.5$ and reference position $\alpha_0 = 269.4481674047229°$, $\delta_0 = +4.743738338140268°$. Thus

$$[\,\boldsymbol{p}_0\;\boldsymbol{q}_0\;\boldsymbol{r}_0\,] = \begin{bmatrix} +0.9999536194300288 & +0.0007964890580278 & -0.0095981557585548 \\ -0.0096311468052894 & +0.0826954601059468 & -0.9965283246797758 \\ +0.0000000000000000 & +0.9965745463752554 & +0.0826992957464197 \end{bmatrix}. \tag{45}$$

In the file, t_B is given in OBMT revolutions (Fig. 5); for the conversion to $\tau = t_\mathrm{B}$ in years the following linear relation between is good to within ± 0.03 s:

$$\tau = (\texttt{tB\_rev} - 5370.125526)/1460.999980\,. \tag{46}$$

The file contains 493 observations with $-2.363957589 \le \tau \le +2.184157454$ yr. The parallax factors f_w and f_z are directly obtained from the file; $f_r = -\Delta t/\tau_\mathrm{A}$, where Δt is the barycentric correction and $\tau_\mathrm{A} = A/c = 499.004783836156$ s.

The least-squares fit must be iterated to handle possible outliers and to take care of the non-linearities of the problem. Using only the AL observations (w) the overdetermined system to solve is

$$\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{x}'}\,\Delta\boldsymbol{x} = \boldsymbol{w}^{(\mathrm{obs})} - \boldsymbol{w}^{(\mathrm{calc})}(\boldsymbol{x})\,, \tag{47}$$

where $\boldsymbol{x}$ is the vector of astrometric parameters, $\boldsymbol{w}^{(\mathrm{obs})}$ the observed AL coordinates from the file, and $\boldsymbol{w}^{(\mathrm{calc})}(\boldsymbol{x})$ the values calculated from Eq. (42). A natural starting approximation would be $\boldsymbol{x} = \boldsymbol{0}$. However, as seen from Eqs. (42)–(44), this will make the partial derivatives with respect to μ_{r0} strictly zero for all observations, leading to a singular system. It is therefore recommended to make first a solution using only the five first parameters in $\boldsymbol{x}$, and enable the sixth parameter later.

For the given data we find that two observations (index 134 and 317) deviate strongly; for the remaining $m = 491$ observations the unweighted 5-parameter solution is:

$$\left.\begin{aligned}
a_0 &= -0.015402 \pm 0.051988 \text{ mas}\\
d_0 &= +0.061399 \pm 0.044729 \text{ mas}\\
\varpi_0 &= +547.281095 \pm 0.067778 \text{ mas}\\
\mu_{\alpha*0} &= -801.732552 \pm 0.041531 \text{ mas yr}^{-1}\\
\mu_{\delta 0} &= +10364.220005 \pm 0.038377 \text{ mas yr}^{-1}
\end{aligned}\right\} \tag{48}$$

The uncertainties were estimated by scaling the inverse of the normal matrix by the unit weight variance $\mathrm{SSR}/(m-5) = 0.519185$ mas^2 (SSR = sum of squared residuals).

Taking the 5-parameter solution (with $\mu_{r0} = 0$) as a starting approximation for an unweighted 6-parameter solution, using the same 491 observations, yields

$$\left.\begin{aligned}
a_0 &= -0.146310 \pm 0.015026 \text{ mas}\\
d_0 &= -0.697899 \pm 0.016470 \text{ mas}\\
\varpi_0 &= +546.910638 \pm 0.020092 \text{ mas}\\
\mu_{\alpha*0} &= -801.669022 \pm 0.011951 \text{ mas yr}^{-1}\\
\mu_{\delta 0} &= +10364.274847 \pm 0.011039 \text{ mas yr}^{-1}\\
\mu_{r0} &= -12990.755622 \pm 176.526506 \text{ mas yr}^{-1}
\end{aligned}\right\} \tag{49}$$

with SSR$/(m-6) = 0.042763$ mas^2. The result for μ_{r0} corresponds to a radial velocity of

$$v_r = A\mu_{r0}/\varpi_0 = -112.60 \pm 1.53 \text{ km s}^{-1} \tag{50}$$

(the uncertainty estimate above takes into account the moderate correlation $\rho = +0.250508$ between the ϖ_0 and μ_{r0}). The effect of the perspective acceleration is highly significant, as shown both by the high signal-to-noise ratio $\simeq 74$ for μ_{r0} and the reduction of the RMS residual from 0.72 mas for the 5-parameter solution to 0.21 mas for the 6-parameter solution.

TABLE 2: A possible specification of input data for object processing based on astrometric elementary observations (e.g. for an astrometric binary). Multiplicity is the total number of times the data item appears for a given object. The flags could code the FOV index and CCDs used in each FOV transit, as well as some quality or reliability information for the observations. Single-precision reals (float) are used whenever there is no risk of compromising accuracy at the 1 μas level. Assuming $N = 80$, the total amount of data is about 4 kByte per object, or 4 TB for all 10^9 objects.
Notes: 1) No photometric data are included. 2) The reference points may possibly be omitted, and taken from the astrometric source catalogue instead. 3) This concept is not applicable to solar-system objects. 4) The data set would be increased by about a factor of 8–9 if some data (w, z, σ_w, σ_z) would be given for each CCD transit individually, instead of averages over one field-of-view transit. In this case, information on the correlation between the different CCD transits would have to added, too (probably just one quantity per field-of-view transit).

Quantity	Designation	Type	Multiplicity	Bytes
Data given once per object:				
identifier	–	long	1	8
reference point in RA	α_0	double	1	8
reference point in Dec	δ_0	double	1	8
number of FOV transits	N	int	1	4
Data given once per FOV transit:				
time (TCB referred to the SSB)	t_{B}	long	N	$8N$
flags	–	int	N	$4N$
position angle of scan	θ	double	N	$8N$
parallax factor AL	f_w	float	N	$4N$
parallax factor AC	f_z	float	N	$4N$
local coordinate AL of image centroid	w	double	N	$8N$
local coordinate AC of image centroid	z	float	N	$4N$
standard error AL	σ_w	float	N	$4N$
standard error AC	σ_z	float	N	$4N$

Total size = $48N + 28$ bytes.

Table 3: A possible specification of input data for object processing and 2-d imaging based on samples in rectangular windows (e.g., for a partially resolved binary). The flags could code the FOV index and CCDs used in each transit, as well as some quality or reliability information for the observations. Assuming $N = 80$, $M = 10$ and window size $(I, L) = (6, 1)$ (faint stars), the total amount of data is about 45 kByte per object, or 45 TB for all 10^9 objects.
Notes: 1)–3) as for Table 2. 4) The sample sizes and shear terms may significantly change over a field-of-view transit. Therefore they are given for each CCD transit separately. 5) The position angle of scan is given only once per field-of-view transit. It is a formal quantity relating the local scan coordinates to the ICRS. Small changes of the actual position angle of scan are of no relevance therefore. Still, in order to avoid any systematic effects, θ should be chosen as the actual position angle of scan at the mid-time of a field-of-view transit, i.e. at the readout time for AF4. Similarly, the parallax factors are given only once, since they remain constant over a minute of time, even for the closest stars.

Quantity	Designation	Type	Multiplicity	Bytes
Data given once per object:				
identifier	–	long	1	8
reference point in RA	α_0	double	1	8
reference point in Dec	δ_0	double	1	8
number of FOV transits	N	int	1	4
Data given once per FOV transit:				
time (TCB referred to the SSB)	t_B	long	N	$8N$
flags	–	int	N	$4N$
position angle of scan	θ	double	N	$8N$
parallax factor AL	f_w	float	N	$4N$
parallax factor AC	f_z	float	N	$4N$
number of samples AL	I	int	N	$4N$
number of samples AC	L	int	N	$4N$
number of CCD transits	M	int	N	$4N$
Data given once per CCD transit:				
local coordinate AL of sample (0,0)	w_0	double	NM	$8NM$
local coordinate AC of sample (0,0)	z_0	float	NM	$4NM$
sample size AL	Δw	float	NM	$4NM$
sample size AC	Δz	float	NM	$4NM$
shear term AL	c_w	float	NM	$4NM$
shear term AC	v_z	float	NM	$4NM$
sample values	S_{il}	float	$NMIL$	$4NMIL$

Total size $= 4NMIL + 28NM + 40N + 28$ bytes.

TABLE 4: A possible specification of input data for object processing and 2-d imaging containing both astrometric elementary observations and window sample data. The flags could code the FOV index and CCDs used in each transit, as well as some quality or reliability information for the observations. Assuming $N = 80$, $M = 10$ and window size $(I, L) = (6, 1)$ (faint stars), the total amount of data is about 46 kByte per object, or 46 TB for all 10^9 objects.

Quantity	Designation	Type	Multiplicity	Bytes
Data given once per object:				
identifier	–	long	1	8
reference point in RA	α_0	double	1	8
reference point in Dec	δ_0	double	1	8
number of FOV transits	N	int	1	4
Data given once per FOV transit:				
time	t	long	N	$8N$
flags	–	int	N	$4N$
position angle of scan	θ	double	N	$8N$
parallax factor AL	f_w	float	N	$4N$
parallax factor AC	f_z	float	N	$4N$
local coordinate AL of image centroid	w	double	N	$8N$
local coordinate AC of image centroid	z	float	N	$4N$
standard error AL	σ_w	float	N	$4N$
standard error AC	σ_z	float	N	$4N$
number of samples AL	I	int	N	$4N$
number of samples AC	L	int	N	$4N$
number of CCD transits	M	int	N	$4N$
Data given once per CCD transit:				
local coordinate AL of sample (0,0)	w_0	double	NM	$8NM$
local coordinate AC of sample (0,0)	z_0	float	NM	$4NM$
sample size AL	Δw	float	NM	$4NM$
sample size AC	Δz	float	NM	$4NM$
shear term AL	c_w	float	NM	$4NM$
shear term AC	v_z	float	NM	$4NM$
sample values	S_{il}	float	$NMIL$	$4NMIL$

Total size $= 4NMIL + 28NM + 60N + 28$ bytes.